Linear Algebra I
Exercise Sheet IV: Eigenspaces

Terminology

Consider a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

How many eigenvectors does this have? In fact, there are uncountably many! There are uncountably many vectors

$$v = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

satisfying $Mv = \lambda v$. In class I speak mostly about *equivalence classes of eigenvectors up to scaling*. In this case we consider all the eigenvectors v to be part of the same class. This is precisely the eigenline. In this exercise sheet, the term *eigenvector* is taken implicitly to mean eigenline.

Following this, the span of eigenvectors is all the points described by the collection of eigenlines. One can also consider the span of eigenlines corresponding to the *same* eigenvalue. This object is called an eigenbasis in the literature. The span of all eigenlines is then made up of the eigenbases.

Eigenbases

Question 1. Compare the spans of eigenlines of the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Compare now the *eigenspaces* of each matrix.

Question 2. Recall that the characteristic polynomial of a 2×2 matrix A is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0.$$

Find a similar expression for a 3×3 matrix.

Question 3. Find the eigenspaces (eigenvalues and eigenvectors) of the following linear maps. Find the span of the eigenvectors and in each case decide whether the range has an eigenbasis.

$$A = \begin{pmatrix} 1 & 2 \\ 6 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 6 & -2 & 1 \\ 1 & 16 & -8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -7 & -3 \\ -2 & -1 \end{pmatrix},$$

Coordinate transformations

Question 4. Does the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ represented by

$$\begin{pmatrix} 1 & 5 & 10 \\ 2 & 3 & 9 \\ -1 & 1 & 0 \end{pmatrix}$$

have an eigenbasis? Sketch the eigenlines of L. What about the linear map $L : \mathbb{C}^3 \to \mathbb{C}^3$ given by the same matrix? Can you sketch the eigenlines of L in this case?

Question 5. Find the representation of the linear maps given by the following matrices in their eigenbases (i.e. diagonalise these matrices)

$$A = \begin{pmatrix} 1 & 2 \\ 6 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 6 & -2 & 1 \\ 1 & 16 & -8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -7 & -3 \\ -2 & -1 \end{pmatrix},$$

Find A-D to the powers 7, 13, 24, and -6 respectively.